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THE NORMED AND BANACH ENVELOPES OF WEAKL^1

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ABSTRACT

The space WeakL^1 consists of all Lebesgue measurable functions on $[0,1]$ such that

$$q(f) = \sup_{c>0} c\,\lambda\{t : |f(t)| > c\}$$

is finite, where λ denotes Lebesgue measure. Let ρ be the gauge functional of the convex hull of the unit ball $\{f : q(f) \leq 1\}$ of the quasi-norm q, and let N be the null space of ρ. The normed envelope of WeakL^1, which we denote by W, is the space (Weak$L^1/N, \rho$). The Banach envelope of WeakL^1, $\overline{W}$, is the completion of W. We show that $\overline{W}$ is isometrically lattice isomorphic to a sublattice of W. It is also shown that all rearrangement invariant Banach function spaces are isometrically lattice isomorphic to a sublattice of W.

Introduction

Let (Ω, Σ, μ) be a measure space. The space Weak$L^1(\mu)$ consists of all (equivalence classes of almost everywhere equal) real-valued Σ-measurable functions f for which the quasinorm

$$q(f) = \sup_{c>0} c\,\mu\{\omega : |f(\omega)| > c\}$$

is finite. This space arose in connection with certain interpolation results, and is of importance in harmonic analysis. If (Ω, Σ, μ) is purely non-atomic, the maximal seminorm ρ on Weak$L^1(\mu)$ was found in [1] and [2] to be

$$\rho(f) = \lim_{n\to\infty} \sup_{\substack{q/p>n \\ p,q>0}} \int_{p\leq|f|\leq q} |f|\,d\mu \,/\, \ln(q/p).$$

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The normed envelope of Weak$L^1(\mu)$ is the normed space

$$W(\mu) = (\text{Weak}L^1(\mu)/N, \rho),$$

where N denotes the null space of the functional ρ. The Banach envelope is the completion $\overline{W(\mu)}$ of $W(\mu)$. In this paper, we consider (up to measure isomorphism) only the measure space $[0,1]$ endowed with Lebesgue measure λ. We denote $W(\lambda)$ and $\overline{W(\lambda)}$ by W and $\overline{W}$ respectively. Peck and Talagrand [6] showed that $\overline{W}$ is universal for the class of all separable Banach lattices with order continuous norm. Recently, Lotz and Peck [5] showed that $\overline{W}$ contains isometrically lattice isomorphic copies of certain sublattices of $\ell^\infty(L^1)$. (Here and throughout, L^1 means $L^1[0,1]$, up to isometric lattice isomorphism.) From this, they deduced that every separable Banach lattice is isometrically lattice isomorphic to a sublattice of $\overline{W}$. In this article, we show that there is a sublattice G of $\ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ such that G, W, and $\overline{W}$ mutually isometrically lattice isomorphically embed in one another. It is also shown that all rearrangement invariant Banach function spaces in the sense of [5] are isometrically lattice isomorphic to sublattices of W. For further results regarding the structure of Weak$L^1(\mu)$, we refer the reader to [3]. Unexplained notation and terminology on vector lattices can be found in [7]. If E is a Banach lattice and I is an arbitrary index set, let $\ell^p(I,E)$, $1 \le p \le \infty$, respectively, $c_0(I,E)$, be the space consisting of all families $(x_i)_{i\in I}$ such that $x_i \in E$ for all i, and $(\|x_i\|)_{i\in I} \in \ell^p(I)$, respectively, $c_0(I)$. We write $\ell^p(E)$ and $c_0(E)$ for these respective spaces if the index set $I = \mathbb{N}$. Clearly $\ell^p(I,E)$ and $c_0(I,E)$ are Banach lattices. The cardinality of a set A is denoted by $|A|$.

1. The spaces W and $\overline{W}$

If f is a real-valued function defined on a set Ω, let the **support** of f be the set $\operatorname{supp} f = \{\omega \in \Omega : |f(\omega)| > 0\}$. Furthermore, for real numbers $p \le q$, we write $\{p \le f \le q\}$ for the set $\{\omega \in \Omega : p \le f(\omega) \le q\}$.

LEMMA 1: *Let (h_k) be a sequence of disjointly supported Lebesgue measurable functions on $[0,1]^2$. Suppose there exist $\delta, \gamma > 0$ and strictly positive sequences (α_k), (β_k) such that*

1. $\alpha_k < \beta_k < \alpha_{k+1}$ *for all* k,
2. $\lim_k \alpha_k = \lim_k \beta_k = \infty$,
3. $\ln(\alpha_{k+1}/\beta_k) \ge (k+1)\sum_{j=1}^{k} \int h_j$ *for all* k, *and*
4. $\delta\alpha_k \le h_k(s,t) \le \gamma\beta_k$ *for all* $(s,t) \in \operatorname{supp} h_k$.

If $1 \le p < q < \infty$, $q/p > \delta\alpha_N$, *and* h *denotes the pointwise sum* $\sum h_k$, *then*

$$\int_{p\le h\le q} h \le \frac{1}{N}\ln\frac{\gamma q}{\delta p} + \sup_k \int_{p\le h_k\le q} h_k.$$

Proof: If $[\delta\alpha_k, \gamma\beta_k] \cap [p,q] = \emptyset$, $\int_{p\le h_k\le q} h_k = 0$. So we may assume that the said intersection is non-empty for some k. Since $\delta\alpha_k \to \infty$, $[\delta\alpha_k, \gamma\beta_k] \cap [p,q] \neq \emptyset$ for at most finitely many k. Let m and n be the minimum and maximum of the set $\{k : [\delta\alpha_k, \gamma\beta_k] \cap [p,q] \neq \emptyset\}$ respectively. We consider two cases.

CASE 1: $m = n$.
In this case,

$$\int_{p\le h\le q} h = \int_{p\le h_m\le q} h_m \le \sup_k \int_{p\le h_k\le q} h_k.$$

CASE 2: $m < n$.
Note that $p \le \gamma\beta_m$, and $q \ge \delta\alpha_n$. Therefore,

$$\ln\frac{\gamma q}{\delta p} \ge \ln\frac{\alpha_n}{\beta_{n-1}} \ge n\sum_{k=1}^{n-1}\int h_k.$$

Now $q > \delta p\alpha_N \ge \delta\alpha_N$; hence $n \ge N$. Thus

$$\begin{aligned}\int_{p\le h\le q} h &= \sum_{k=m}^{n-1}\int_{p\le h_k\le q} h_k + \int_{p\le h_n\le q} h_n\\ &\le \sum_{k=1}^{n-1}\int h_k + \int_{p\le h_n\le q} h_n\\ &\le \frac{1}{n}\ln\frac{\gamma q}{\delta p} + \sup_k\int_{p\le h_k\le q} h_k\\ &\le \frac{1}{N}\ln\frac{\gamma q}{\delta p} + \sup_k\int_{p\le h_k\le q} h_k. \qquad \blacksquare\end{aligned}$$

Write any element $g \in \ell^\infty(\ell^\infty(L^1))$ as $g = (g_{ij})$, where $g_{ij} \in L^1$ for all i, j, and $\sup_{i,j}\|g_{ij}\|_{L^1} < \infty$. For any double sequence of numbers $M = (M_{ij})$ such that $M_{ij} \ge 1$ for all i, j, let $F = F_M$ be the (non-closed) lattice ideal of $\ell^\infty(\ell^\infty(L^1))$ consisting of all $g = (g_{ij}) \in \ell^\infty(\ell^\infty(L^1))$ such that $\sup_{i,j}\|g_{ij}\|_{L^\infty}/M_{ij} < \infty$. For the next result, we follow the idea of Lotz and Peck [5] in considering the WeakL^1 space defined on the unit square $[0,1]^2$ endowed with Lebesgue measure. Since $[0,1]$ and $[0,1]^2$ are isomorphic measure spaces, their corresponding WeakL^1 spaces are isometrically lattice isomorphic; the same holds for the respective normed and Banach envelopes.

PROPOSITION 2: *There exists a lattice homomorphism $T : F \to W$ of norm ≤ 1 which vanishes on $F \cap c_0(\ell^\infty(L^1))$.*

Proof: Choose positive sequences (ε_n) and (r_i) with limits 0 and ∞ respectively so that $r_i > 1 \geq \varepsilon_n$ for all i and n. For each n, let E_n be the conditional expectation operator on L^1 with respect to the σ-algebra generated by $\{[\frac{m-1}{2^n}, \frac{m}{2^n}) : 1 \leq m \leq 2^n\}$. If $i, j, n \in \mathbb{N}$, let A_{ijn} be a countable set which is dense in

$$\{f \in E_n L^1 : \|f\|_{L^1} = 1, \quad \varepsilon_n \leq f \leq nM_{ij}\}$$

with respect to the L^∞-norm. For each $f \in A_{ijn}$, let $(a_m(f))_{m=1}^{2^n}$ be the coefficients such that

$$f = \sum_{m=1}^{2^n} a_m(f)\chi_{[(m-1)/2^n, m/2^n)}.$$

Note that $\varepsilon_n \leq a_m(f) \leq 2^n$ for $1 \leq m \leq 2^n$. Arrange $\bigcup A_{ijn}$ into a sequence (f_k). For each k, determine $i(k), j(k), n(k)$ such that $f_k \in A_{i(k),j(k),n(k)}$. Choose a positive sequence (b_k) so that if we define $\alpha_k = b_k/2^{n(k)}$, and $\beta_k = M_{i(k),j(k)} r_{i(k)} b_k/\varepsilon_{n(k)}$, then $\alpha_k < \beta_k < \alpha_{k+1}$, $\lim_k \alpha_k = \infty = \lim_k \beta_k$, and

$$\ln \frac{\alpha_{k+1}}{\beta_k} \geq (k+1) \sum_{l=1}^{k} \ln r_{i(l)}.$$

Let $g = (g_{ij}) \in F$, and $k \in \mathbb{N}$. Define a function h_k on $[0,1]^2$ by

$$h_k(s,t) = \sum_{m=1}^{2^{n(k)}} \frac{g_{i(k),j(k)}(t)}{s} \chi_{B_{km}},$$

where

$$B_{km} = \left\{(s,t) : \frac{a_m(f_k)}{r_{i(k)} b_k} < s < \frac{a_m(f_k)}{b_k}, \frac{m-1}{2^{n(k)}} < t < \frac{m}{2^{n(k)}}\right\}.$$

The map S defined by $Sg = \sum h_k$, where the sum is taken pointwise, is a linear map from F into the space of Lebesgue measurable functions on $[0,1]^2$. Notice that

$$\begin{aligned}
\operatorname{supp} h_k &\subseteq \bigcup_{m=1}^{2^{n(k)}} \left\{(s,t) : \frac{a_m(f_k)}{r_{i(k)} b_k} < s < \frac{a_m(f_k)}{b_k}\right\} \\
&\subseteq \left\{(s,t) : \frac{\varepsilon_{n(k)}}{r_{i(k)} b_k} < s < \frac{2^{n(k)}}{b_k}\right\} \\
&\subseteq \left\{(s,t) : \frac{1}{\beta_k} < s < \frac{1}{\alpha_k}\right\}.
\end{aligned}$$

Hence the h_k's are pairwise disjoint. As the sets $B_{km}, 1 \le m \le 2^{n(k)}$, are also pairwise disjoint for each k, it follows readily that S is a lattice homomorphism. Suppose $g \in F$, $\|g\| = \sup_{i,j} \|g_{ij}\|_{L^1} \le 1$, let us estimate the ρ-norm of the function Sg. In the first instance, let us assume additionally that there exists $\delta > 0$ such that $g_{ij}(t) \ge \delta$ for all i, j, and t. Set $\gamma = \sup_{i,j} \|g_{ij}\|_{L^\infty}/M_{ij}$. If $(s,t) \in \operatorname{supp} h_k$, then

$$\frac{\delta}{s} \le \frac{g_{i(k),j(k)}(t)}{s} = h_k(s,t) \le \frac{\gamma M_{i(k),j(k)}}{s},$$

and

$$\frac{M_{i(k),j(k)}}{\beta_k} = \frac{\varepsilon_{n(k)}}{r_{i(k)} b_k} < s < \frac{2^{n(k)}}{b_k} = \frac{1}{\alpha_k}.$$

Hence

$$\delta\alpha_k \le h_k(s,t) \le \gamma\beta_k.$$

Moreover,

$$\begin{aligned} \int h_k &= \sum_{m=1}^{2^{n(k)}} \int_{\frac{m-1}{2^{n(k)}}}^{\frac{m}{2^{n(k)}}} \int_{\frac{a_m(f_k)}{r_{i(k)} b_k}}^{\frac{a_m(f_k)}{b_k}} \frac{g_{i(k),j(k)}(t)}{s}\, ds\, dt \\ &= \sum_{m=1}^{2^{n(k)}} \int_{\frac{m-1}{2^{n(k)}}}^{\frac{m}{2^{n(k)}}} g_{i(k),j(k)}(t)\, dt\, \ln r_{i(k)} \qquad (1) \\ &= \|g_{i(k),j(k)}\|_{L^1} \ln r_{i(k)} \le \ln r_{i(k)}. \end{aligned}$$

Therefore,

$$\ln \frac{\alpha_{k+1}}{\beta_k} \ge (k+1) \sum_{l=1}^{k} \ln r_{i(l)} \ge (k+1) \sum_{l=1}^{k} \int h_l.$$

By Lemma 1, if $q/p > \delta\alpha_N$, and $p \ge 1$, then

$$\int_{p \le Sg \le q} Sg \le \frac{1}{N} \ln \frac{\gamma q}{\delta p} + \sup_k \int_{p \le h_k \le q} h_k.$$

If $q/p > \delta\alpha_N$ and $0 < p < 1$, then, using Lemma 1 again,

$$\int_{p \le Sg \le q} Sg \le \int_{p \le Sg \le 1} Sg + \int_{1 \le Sg \le q/p} Sg \le 1 + \frac{1}{N} \ln \frac{\gamma q}{\delta p} + \sup_k \int_{1 \le h_k \le q/p} h_k.$$

Hence

$$\lim_{n\to\infty} \sup_{\substack{q/p>n \\ p,q>0}} \int_{p \le Sg \le q} Sg/\ln(q/p) \le \lim_{n\to\infty} \sup_{\substack{q/p>n \\ p,q>0}} \sup_k \int_{p \le h_k \le q} h_k/\ln(q/p). \qquad (2)$$

Now

$$\int_{p\le h_k\le q} h_k \le \int_0^1 \int_{g_{i(k),j(k)}(t)/q}^{g_{i(k),j(k)}(t)/p} \frac{g_{i(k),j(k)}(t)}{s}\,ds\,dt$$
$$= \|g_{i(k),j(k)}\|_{L^1} \ln\frac{q}{p} \le \ln\frac{q}{p}.$$

Therefore, equation (2) implies that $\rho(Sg) \le 1$. For a general $g = (g_{ij}) \in F$, and any $\delta > 0$, let $g' = (g'_{ij})$, where $g'_{ij} = |g_{ij}| + \delta$. By the above calculation, $\rho(Sg') \le \|g'\| = \|g\| + \delta$. Since S is a lattice homomorphism, $|Sg'| \ge |Sg|$. Thus $\rho(Sg) \le \rho(Sg') \le \|g\| + \delta$. As $\delta > 0$ is arbitrary, we conclude that $\rho(Sg) \le \|g\|$. In particular, applying Lemma 1 in [5], we see that S maps into WeakL^1. It is now clear that the map $T : F \to W$ defined by $Tg = Sg + N$ is a lattice homomorphism of norm ≤ 1.

It remains to show that T vanishes on $F \cap c_0(\ell^\infty(L^1))$. By the continuity of T, it suffices to show that $Tg = 0$ for all $g = (g_{ij}) \in F$ such that there exists $i_0 \in \mathbb{N}$ with $g_{ij} = 0$ whenever $i \ne i_0$. As above, we may assume additionally that there exists $\delta > 0$ such that $g_{i_0 j}(t) > \delta$ for all j and t. If $h_k \ne 0$, then $g_{i(k),j(k)} \ne 0$; hence $i(k) = i_0$. Using (1),

$$\int_{p\le h_k\le q} h_k \le \int h_k \le \|g\| \ln r_{i(k)} = \|g\| \ln r_{i_0}.$$

By (2),

$$\rho(Sg) \le \lim_{n\to\infty} \sup_{\substack{q/p>n\\ p,q>0}} \frac{\|g\| \ln r_{i_0}}{\ln(q/p)} = 0. \qquad \blacksquare$$

Let $Q : \ell^\infty(\ell^\infty(L^1)) \to \ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ be the quotient map. Since Q is a lattice homomorphism, $G = QF$ is a sublattice of $\ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$.

THEOREM 3: *There exists an isometric lattice isomorphism from QF into W.*

Proof: Let T be the map defined in the proof of Proposition 2. Since T vanishes on $F \cap c_0(\ell^\infty(L^1))$, there exists $R : QF \to W$ such that $T = RQ_{|F}$. Now R is a lattice homomorphism, since both T and Q are, and $\|R\| \le \|T\| \le 1$. We claim that $\rho(RQg) \ge \|Qg\|$ for all $g \in F$. Suppose $g = (g_{ij}) \in F$, and $\|Qg\| = 1$. We may assume that there exist sequences of natural numbers $(i'(l))$, $(j'(l))$ such that $(i'(l))$ increases to ∞, and $\|g_{i'(l),j'(l)}\|_{L^1} = 1$ for all l. Recall the sequence (f_k) chosen in the proof of Proposition 2. Given $\eta > 0$, there exists a sequence $(k(l))$ in $\mathbb{N}$ such that $f_{k(l)} \in \bigcup_n A_{i'(l),j'(l),n}$,

$$\sup_l \||g_{i'(l),j'(l)}| - f_{k(l)}\|_{L^1} \le \eta \quad\text{and}\quad \sup_l \frac{\|f_{k(l)}\|_{L^\infty}}{M_{i'(l),j'(l)}} < \infty.$$

Let $\phi_{ij} = f_{k(l)}$, and $\psi_{ij} = g_{i'(l),j'(l)}$ if $(i,j) = (i'(l), j'(l))$, $l \in \mathbb{N}$, and $\phi_{ij} = \psi_{ij} = 0$ otherwise. Then $\phi = (\phi_{ij})$ and $\psi = (\psi_{ij})$ are both in F, and $\|\phi - |\psi|\| \leq \eta$. Since $\|T\| \leq 1$,

$$\rho(T\psi) = \rho(T|\psi|) \geq \rho(T\phi) - \eta.$$

Then

$$|g| \geq \psi \Longrightarrow |Tg| \geq T\psi \Longrightarrow \rho(Tg) \geq \rho(T\psi) \geq \rho(T\phi) - \eta.$$

For a given l, write $f_{k(l)} = \sum_{m=1}^{2^n} a_m \chi_{[(m-1)/2^n, m/2^n)}$ for some $(a_m)_{m=1}^{2^n}$, and some n. Note that $i(k(l)) = i'(l)$, $j(k(l)) = j'(l)$, and $n(k(l)) = n$. By definition of T, for $1 \leq m \leq 2^n$, $(s,t) \in B_{k(l),m}$, $|T\phi(s,t)| = a_m/s$. In particular, $b_{k(l)} < |T\phi(s,t)| < r_{i'(l)} b_{k(l)}$ for $(s,t) \in \cup_{m=1}^{2^n} B_{k(l),m}$. Therefore,

$$\begin{aligned}\int_{b_{k(l)} \leq |T\phi| \leq r_{i'(l)} b_{k(l)}} |T\phi| &\geq \sum_{m=1}^{2^n} \iint_{B_{k(l),m}} \frac{a_m}{s}\, ds\, dt \\ &= \sum_{m=1}^{2^n} \frac{a_m}{2^n} \ln r_{i'(l)} = \|f_{k(l)}\|_{L^1} \ln r_{i'(l)}.\end{aligned}$$

Since $\lim_l r_{i'(l)} = \infty$, we see that $\rho(T\phi) \geq \limsup_l \|f_{k(l)}\|_{L^1} \geq 1 - \eta$. As $\eta > 0$ is arbitrary, it follows immediately that $\rho(RQg) = \rho(Tg) \geq 1$. ∎

Observe that if $M = (M_{ij})$ and $M' = (M'_{ij})$ satisfy $\sup_j M_{ij} = \sup_j M'_{ij} = \infty$ for all i, then each of QF_M and $QF_{M'}$ is isometrically lattice isomorphic to a sublattice of the other. For the remainder of this section, let

$$M_{ij} = (i+1)j/\ln(i+1) \quad \text{for all } i,j \in \mathbb{N}.$$

The next result and Theorem 3 together show that $QF = QF_M$ is a maximal sublattice of W.

THEOREM 4: *There is an isometric lattice isomorphism from W into QF.*

Proof: Given $f \in \text{Weak}L^1$, let $g_{ij} = f\chi_{\{j \leq |f| \leq (i+1)j\}}/\ln(i+1)$ for all $i,j \in \mathbb{N}$. It is easy to see that $g = (g_{ij}) \in F$, and that

$$\|Qg\| = \limsup_{i\to\infty} \sup_j \|g_{ij}\|_{L^1} = \rho(f). \tag{3}$$

Consider the mapping L: $\text{Weak}L^1 \to QF$ defined by $Lf = Qg$. It follows from the proof of the Key Lemma 2.3 in [3] that L is linear. Now (3) tells us that the map $\tilde{L}$: $W \to QF$, $\tilde{L}(f+N) = Lf$, is well defined and a linear isometry. Also,

$$\tilde{L}(|f+N|) = L|f| = Q|g| = |Qg| = |Lf| = |\tilde{L}(f+N)|.$$

Hence $\tilde{L}$ is the isometric lattice isomorphism sought. ∎

THEOREM 5: *There exists an isometric lattice isomorphism from $\overline{W}$ into W.*

Proof: It is easily verified that the set

$$D = \{Qg : g = (g_{ij}) \in \ell^\infty(\ell^\infty(L^1)), \quad \|g_{ij}\|_{L^\infty} \le M_{ij} \quad \text{for all } i, j\}$$

is closed in $\ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$. Let $\tilde{L}$: $W \to QF$ be the isometric lattice isomorphism given in Theorem 4. By definition of $\tilde{L}$, $\tilde{L}(W) \subseteq D$. Now there is a unique continuous linear extension $L^\#$: $\overline{W} \to \ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ of $\tilde{L}$. Since $\tilde{L}(W) \subseteq D$, and D is closed, $L^\#(\overline{W}) \subseteq D \subseteq QF$. Obviously, $L^\#$ is an isometric lattice isomorphism. Let R: $QF \to W$ be the isometric lattice isomorphism constructed in Theorem 3, then $RL^\#$ is an isometric lattice isomorphism from $\overline{W}$ into W. ∎

2. Rearrangement invariant spaces

In this section, we show that if E is a rearrangement invariant space in the sense of [4, §2a], then E is isometrically lattice isomorphic to a sublattice of W. The result is inspired by Theorem 4 in [5], where it was shown that the WeakL^p spaces defined on separable measure spaces are isometrically lattice isomorphic to sublattices of $\overline{W}$. We provide the proof only for the rearrangement invariant spaces defined on $[0, \infty)$. The proofs for the measure spaces $[0, 1]$ and $\mathbb{N}$ can be obtained by making some obvious adjustments. Recall that if E is a rearrangement invariant space (or, more generally, a Köthe function space [4, Definition 1.b.17]), every measurable function h such that hf is integrable for all $f \in E$ defines a bounded linear functional x'_h on E by $x'_h(f) = \int fh$. Such functionals are called **integrals**. Every simple function generates an integral on E.

LEMMA 6: *Let E be a rearrangement invariant space on $[0, \infty)$. There exists a sequence of simple functions (h_i) such that $\|x'_{h_i}\| \le 1$ for all n, and $\|f\| = \limsup_{i\to\infty} |\int fh_i|$ for all $f \in E$.*

Proof: Let $\mathcal{F}$ be the collection of all simple functions of the form

$$h = \sum_{j=1}^{k} a_j \chi_{[c_{j-1}, c_j)},$$

where $k \in \mathbb{N}$, $(a_j)_{j=1}^k$, $(c_j)_{j=0}^k$ are rational numbers, and $0 = c_0 < c_1 < \cdots < c_k$. Define $\mathcal{F}_1$ to be the subset $\{h \in \mathcal{F} : \|x'_h\| \le 1\}$. We claim that for any $f \in E$, and any $\varepsilon > 0$, there exists $h \in \mathcal{F}_1$ such that $|\int fh| > \|f\| - \varepsilon$. Let $f \in E$ and $\varepsilon > 0$ be given. By definition of rearrangement invariant spaces, there exists an integral $x'_g \in E'$ such that $\|x'_g\| \leq 1$, and $|x'_g(f)| = |\int fg| > \|f\| - \varepsilon/2$. Let (g_n) be a sequence of simple functions which converges to g pointwise, and such that $|g_n| \leq |g|$ for all n. By the Lebesgue Dominated Convergence Theorem, $\lim_n \int fg_n = \int fg$. We may thus assume additionally that g is a simple function. It is easy to see that there exists $h \in \mathcal{F}$ such that $|\int fh| \geq |\int fg| - \varepsilon/2 > \|f\| - \varepsilon$, and that $h^* \leq g^*$, where h^* and g^* are the decreasing rearrangements of $|h|$ and $|g|$ respectively. Thus $\|x'_h\| \leq \|x'_g\| \leq 1$. Therefore, $h \in \mathcal{F}_1$, as desired.

Since $\mathcal{F}_1$ is countable, we can arrange for a sequence (h_i) so that each element of $\mathcal{F}_1$ appears infinitely many times in the sequence. Clearly the sequence (h_i) fulfills the conditions of the lemma. ∎

THEOREM 7: *Every rearrangement invariant space E on $[0,\infty)$ is isometrically lattice isomorphic to a sublattice of W.*

Proof: We will show that E is isometrically lattice isomorphic to a sublattice of $\overline{QF_M}$ for some suitably chosen double sequence $M = (M_{ij})$. Then, by Theorem 3, E is isometrically lattice isomorphic to a sublattice of $\overline{W}$, which in turn is isometrically lattice isomorphic to a sublattice of W by Theorem 5.

Let (h_i) be the sequence given by Lemma 6. Since h_i is a simple function, there exists $0 < a_i < \infty$ such that $\operatorname{supp} h_i \subseteq [0, a_i]$. For $f \in E$, $i \in \mathbb{N}$, and $t \in [0,1]$, define $f_{i1}(t) = a_i f(a_i t)|h_i(a_i t)|$. Also let $f_{ij} = 0$ for all $i \in \mathbb{N}$ and all $j > 1$. Clearly

$$\|f_{i1}\|_{L^1} = \int_0^{a_i} |f(u)h_i(u)|\,du = \int_0^\infty |f(u)h_i(u)|\,du. \tag{4}$$

Thus $\|f_{i1}\|_{L^1} \leq \|f\| \cdot \|x'_{|h_i|}\| = \|f\| \cdot \|x'_{h_i}\| \leq \|f\|$ for all i. Hence $(f_{ij}) \in \ell^\infty(\ell^\infty(L^1))$. The map $T\colon E \to \ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ defined by $Tf = Q(f_{ij})$ is easily seen to be a lattice homomorphism. It follows from the preceding calculation that $\|T\| \leq 1$. On the other hand, by equation (4),

$$\limsup_i \sup_j \|f_{ij}\|_{L^1} = \limsup_i \|f_{i1}\|_{L^1} = \limsup_i \int |fh_i| \geq \limsup_i \left|\int fh_i\right| \geq \|f\|.$$

Therefore, T is an isometry. To complete the proof, it suffices to produce a sequence (M_i) such that $\lim_i \|f_{i1}\chi_{\{|f_{i1}|>M_i\}}\|_{L^1} = 0$. For then, if we define $M_{i1} = \max\{M_i, 1\}$, and $M_{ij} = 1$ for $j > 1$, it is easy to check that $TE \subseteq \overline{QF_M}$, where $M = (M_{ij})$.

Let $K_i = \|h_i\|_{L^\infty}$ for all i. First note that for $f \in E$, $\|f\| \geq \int_0^1 f^*(t)\,dt$; hence $c\lambda\{|f| > c\} \leq \|f\|$ if $c > \|f\|$. Therefore, if $c > a_i K_i \|f\|$,

$$\begin{aligned} \lambda\{|f_{i1}| > c\} &\leq \lambda\Big\{t : |f(a_i t)| > \frac{c}{a_i K_i}\Big\} \\ &= \frac{1}{a_i}\lambda\Big\{|f| > \frac{c}{a_i K_i}\Big\} \\ &\leq \frac{K_i}{c}\|f\|. \end{aligned} \tag{5}$$

CASE 1: $\sup_i K_i = K < \infty$

Let (M_i) be any sequence such that $M_i/a_i \uparrow \infty$. Let $f \in E$. For all i such that $M_i > a_i K\|f\|$, $\lambda\{|f_{i1}| > M_i\} \leq K\|f\|/M_i$ by (5). Hence

$$\begin{aligned} \|f_{i1}\chi_{\{|f_{i1}|>M_i\}}\|_{L^1} &\leq \int_0^{K\|f\|/M_i} f_{i1}^*(t)\,dt \\ &\leq \int_0^{Ka_i\|f\|/M_i} f^*(t)h_i^*(t)\,dt \\ &\leq K\int_0^{Ka_i\|f\|/M_i} f^*(t)\,dt. \end{aligned}$$

Since $\int_0^1 f^*(t)\,dt \leq \|f\| < \infty$, we obtain that $\lim_i \|f_{i1}\chi_{\{|f_{i1}|>M_i\}}\|_{L^1} = 0$.

CASE 2: $\sup_i K_i = \infty$

For each i, choose $b_i > 0$ such that $h_i^*(b_i) \geq K_i/2$. Then, for all $f \in E$,

$$\frac{K_i}{2}\int_0^{b_i} f^*(t)\,dt \leq \int_0^{b_i} f^*(t)h_i^*(t)\,dt \leq \|f\| \tag{6}$$

since $\|x'_{h_i^*}\| = \|x'_{h_i}\| \leq 1$. Let (n_i) be chosen so that $\lim_i K_i/K_{n_i} = 0$. Now let (M_i) be a sequence such that $(a_iK_i)^{-1}M_i > \max\{i, i/b_{n_i}\}$ for all i. If $f \in E$, and $i > \|f\|$, then $\lambda\{|f_{i1}| > M_i\} \leq K_i\|f\|/M_i$ by (5). Therefore,

$$\begin{aligned} \|f_{i1}\chi_{\{|f_{i1}|>M_i\}}\|_{L^1} &\leq \int_0^{K_i\|f\|/M_i} f_{i1}^*(t)\,dt \\ &\leq K_i\int_0^{a_iK_i\|f\|/M_i} f^*(t)\,dt \\ &\leq K_i\int_0^{b_{n_i}} f^*(t)\,dt \\ &\leq \frac{2K_i\|f\|}{K_{n_i}} \quad \text{by (6)}. \end{aligned}$$

It follows that $\lim_i \|f_{i1}\chi_{\{|f_{i1}|>M_i\}}\|_{L^1} = 0$. ∎

Theorem 7 can be extended to certain rearrangement invariant spaces defined on non-separable measure spaces. Endow the two-point set $\{-1,1\}$ with the measure which assigns a mass of 1/2 to each singleton set. For any index set I, denote by μ the product measure on $\{-1,1\}^I$. If I is countable, $\{-1,1\}^I$ is measure isomorphic to $[0,1]$. For the remainder of this section, fix an index set I which has the cardinality of the continuum. For each $i \in I$, let $\varepsilon_i : \{-1,1\}^I \to \{-1,1\}$ be the projection onto the i-th coordinate. If J is a finite subset of I, and $\delta = (\delta_i)_{i\in J} \in \{-1,1\}^J$, define $\phi_{J,\delta}$ to be the function $\prod_{i\in J} \chi_{\{\varepsilon_i=\delta_i\}}$ on $\{-1,1\}^I$. Let Φ_J be the span of the functions $\{\phi_{J,\delta} : \delta \in \{-1,1\}^J\}$. It is not hard to see that the set $\Phi = \bigcup\{\Phi_J : J \subseteq I, |J| < \infty\}$ is a vector lattice (with the usual pointwise operations and order). Define E by

$$E = \{\mathbf{f} = (f_i)_{i\in I} : f_i \in \Phi \quad \text{for all } i, \quad f_i \neq 0 \quad \text{for at most finitely many } i\}.$$

Similarly, let E_J consist of all $\mathbf{f} = (f_i)_{i\in I} \in E$ such that $f_i \in \Phi_J$ for all i. Then E is a vector lattice with the coordinatewise operations and order, and $E = \bigcup\{E_J : J \subseteq I, |J| < \infty\}$. A norm $\|\cdot\|$ on E is called a **lattice norm** if $|\mathbf{f}| \leq |\mathbf{g}|$ implies $\|\mathbf{f}\| \leq \|\mathbf{g}\|$. For $\mathbf{f} = (f_i) \in E$, let the **distribution function** $d_{\mathbf{f}}$ of $\mathbf{f}$ be defined by $d_{\mathbf{f}}(t) = \sum_{i\in I} \mu\{|f_i| > t\}$, $t \geq 0$.

THEOREM 8: *Let $\|\cdot\|$ be a lattice norm on E which is rearrangement invariant in the sense that $\mathbf{f}, \mathbf{g} \in E$, $d_{\mathbf{f}} = d_{\mathbf{g}}$ implies $\|\mathbf{f}\| = \|\mathbf{g}\|$. Then $(E, \|\cdot\|)$ is isometrically lattice isomorphic to a sublattice of W.*

Of course, it follows that the completion of E, $\overline{E}$, is isometrically isomorphic to a sublattice of $\overline{W}$. Since $\overline{W}$ is isometrically lattice isomorphic to a sublattice of W by Theorem 5, the same is true for $\overline{E}$. This leads immediately to the following corollary.

COROLLARY 9: *If $1 \leq p < \infty$, then $\ell^p(I, L^p(\{-1,1\}^I))$ is isometrically isomorphic to a sublattice of W.*

As indicated above, L^1 may be identified (as a Banach lattice) with $L^1(\{-1,1\}^{\mathbb{Z}})$. This identification will be in force for the rest of the section. For each $k \in \mathbb{Z}$, let r_k: $\{-1,1\}^{\mathbb{Z}} \to \{-1,1\}$ be the projection onto the k-th coordinate. Select a bijection γ: $I \to \{-1,1\}^{\mathbb{N}}$. Thus, for every $i \in I$, $\gamma(i) = (\gamma(i,k))_{k=1}^\infty$, where $\gamma(i,k) = \pm 1$ for all $i \in I$, $k \in \mathbb{N}$. Finally, for every i, pick a strictly decreasing sequence of negative integers $k_i = (k_i(m))_{m=1}^\infty$ such that

- for each m, $\{k_i(m) : i \in I\}$ has only finitely many distinct values;
- if $i \neq i'$, then $\{m : k_i(m) = k_{i'}(m)\}$ is finite.

Given a finite subset J of I, $\delta \in \{-1,1\}^J$, $i \in I$, and $m \in \mathbb{N}$, define, on $\{-1,1\}^{\mathbb{Z}}$,

$$\psi_{J,\delta,i,m} = 2^m \prod_{k=1}^{m} \chi_{\{r_k=\gamma(i,k)\}} \cdot \prod_{j\in J} \chi_{\{r_{k_j(m)}=\delta_j\}}.$$

The mapping $T_{J,m}\colon E_J \to L^1$ is defined by

$$T_{J,m}\mathbf{f} = \sum_{i\in I} \sum_{\delta\in\{-1,1\}^J} a(i,\delta)\psi_{J,\delta,i,m}$$

for all $\mathbf{f} = (f_i)_{i\in I} \in E_J$, where $f_i = \sum_{\delta\in\{-1,1\}^J} a(i,\delta)\phi_{J,\delta}$. Let us remark that the sum over i is in fact a finite sum, since $f_i = 0$ for all but finitely many i. It is clear that $T_{J,m}$ is linear. If I_0 and J are finite subsets of I, there exists $m_0 = m_0(I_0, J) \in \mathbb{N}$ such that

- $(\gamma(i,1),\ldots,\gamma(i,m_0)) \neq (\gamma(i',1),\ldots,\gamma(i',m_0))$ if $i,i' \in I_0$, $i \neq i'$,
- $k_j(m) \neq k_{j'}(m)$ whenever $j,j' \in J$, $j \neq j'$, and $m \geq m_0$.

The following lemma is easily verified by direct computation.

LEMMA 10: *Let I_0, J_1, and J_2 be finite subsets of I such that $J_1 \subseteq J_2$, and let $m \geq m_0(I_0, J_2)$. If*

$$\sum_{\delta\in\{-1,1\}^{J_1}} a(i,\delta)\phi_{J_1,\delta} = \sum_{\eta\in\{-1,1\}^{J_2}} b(i,\eta)\phi_{J_2,\eta}, \quad \text{for all } i \in I_0,$$

or

$$\sum_{i\in I_0} \sum_{\delta\in\{-1,1\}^{J_1}} a(i,\delta)\psi_{J_1,\delta,i,m} = \sum_{i\in I_0} \sum_{\eta\in\{-1,1\}^{J_2}} b(i,\eta)\psi_{J_2,\eta,i,m},$$

then for all $\eta \in \{-1,1\}^{J_2}$, and all $i \in I_0$, $b(i,\eta) = a(i,\delta)$, where $\delta = \eta_{|J_1}$.

An obvious consequence of the lemma is the following proposition.

PROPOSITION 11: *Let I_0, J_1, and J_2 be finite subsets of I such that $J_1 \subseteq J_2$, and let $m \geq m_0(I_0, J_2)$. If $\mathbf{f} = (f_i)_{i\in I} \in E_{J_1}$, and $f_i = 0$ for all $i \notin I_0$, then $T_{J_1,m}\mathbf{f} = T_{J_2,m}\mathbf{f}$.*

For each $\mathbf{f} \in E$, choose a finite subset $J(\mathbf{f})$ of I such that $\mathbf{f} \in E_{J(\mathbf{f})}$. Given a double sequence (h_{mn}) of non-negative measurable functions on $\{-1,1\}^{\mathbb{Z}}$ such that $\sup_{mn} \|T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_{mn}\|_{L^1} < \infty$ for all $\mathbf{f} \in E$, consider the (non-linear) mapping $T : E \to \ell^\infty(\ell^\infty(L^1))$ defined by $T\mathbf{f} = (T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_{mn})_{mn}$.

PROPOSITION 12: *Let Q: $\ell^\infty(\ell^\infty(L^1)) \to \ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ be the quotient map. Then QT is a linear lattice homomorphism.*

Proof: Let $\mathbf{f} = (f_i)_{i\in I}$, $\mathbf{g} = (g_i)_{i\in I} \in E$, and let $\alpha \in \mathbb{R}$. Choose a finite subset I_0 of I such that $f_i = 0 = g_i$ if $i \notin I_0$. Define $J = J(\mathbf{f}) \cup J(\mathbf{g}) \cup J(\alpha\mathbf{f} + \mathbf{g})$. If $m \geq m_0(I_0, J)$, then, for all $n \in \mathbb{N}$,

$$\begin{aligned} T_{J(\alpha\mathbf{f}+\mathbf{g}),m}(\alpha\mathbf{f} + \mathbf{g}) \cdot h_{mn} &= T_{J,m}(\alpha\mathbf{f} + \mathbf{g}) \cdot h_{mn} && \text{by Proposition 11} \\ &= \alpha T_{J,m}\mathbf{f} \cdot h_{mn} + T_{J,m}\mathbf{g} \cdot h_{mn} && \text{by linearity of } T_{J,m} \\ &= \alpha T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_{mn} + T_{J(\mathbf{g}),m}\mathbf{g} \cdot h_{mn} && \text{by Proposition 11.} \end{aligned}$$

Hence QT is linear. Now let $J' = J(\mathbf{f}) \cup J(|\mathbf{f}|)$. Note that the functions $\{\psi_{J',\eta,i,m} : i \in I_0, \eta \in \{-1,1\}^{J'}\}$ are pairwise disjoint if $m \geq m_0(I_0, J')$. Thus $T_{J',m}|\mathbf{f}| = |T_{J',m}\mathbf{f}|$ for all $m \geq m_0(I_0, J')$. For all such m, and all $n \in \mathbb{N}$, it follows from Proposition 11 that

$$|T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_{mn}| = |T_{J(\mathbf{f}),m}\mathbf{f}|\cdot h_{mn} = |T_{J',m}\mathbf{f}|\cdot h_{mn} = T_{J',m}|\mathbf{f}|\cdot h_{mn} = T_{J(|\mathbf{f}|),m}|\mathbf{f}|\cdot h_{mn}.$$

Therefore, $|QT\mathbf{f}| = QT|\mathbf{f}|$, as required. ∎

Given $m \in \mathbb{N}$, the set $K_m = \{k_i(m) : i \in I\}$ is a finite subset of negative integers. Let $K'_m = \{1, 2, \ldots, m\} \cup K_m$. If $\eta = (\eta_k) \in \{-1,1\}^{K'_m}$, let $\zeta_{\eta,m}$ be the function $\prod_{k\in K'_m} \chi_{\{r_k=\eta_k\}}$ defined on $\{-1,1\}^{\mathbb{Z}}$. Associate with each real sequence $c = (c_\eta)_{\eta\in\{-1,1\}^{K'_m}}$ a function $h_c = \sum_{\eta\in\{-1,1\}^{K'_m}} c_\eta\zeta_{\eta,m}$. Also, for each m, choose subsets I_m and J_m of I such that $|I_m| = 2^m$, and $|J_m| = |K_m|$. There exists a bijection $\pi_m\colon I_m \times \{-1,1\}^{J_m} \to \{-1,1\}^{K'_m}$. Given $c = (c_\eta)_{\eta\in\{-1,1\}^{K'_m}}$, define $\mathbf{h}_c = (h_{i,c})_{i\in I}$ by $h_{i,c} = \sum_{\tau\in\{-1,1\}^{J_m}} c_{\pi_m(i,\tau)}\phi_{J_m,\tau}$ for $i \in I_m$, and $h_{i,c} = 0$ otherwise.

LEMMA 13: *Let $\mathbf{f} = (f_i)_{i\in I} \in E$, and let I_0 be a finite subset of I such that $f_i = 0$ if $i \notin I_0$. If $m \geq m_0(I_0, J(\mathbf{f}))$, and $c = (c_\eta)_{\eta\in\{-1,1\}^{K'_m}}$, then there exists $\tilde{\mathbf{h}} = (\tilde{h}_i)_{i\in I}$, such that $d_{\tilde{\mathbf{h}}} = d_{\mathbf{h}_c}$, and $\|T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_c\|_{L^1} = \sum_{i\in I} \int |f_i\tilde{h}_i|$.*

Proof: Write $f_i = \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} a(i,\delta)\phi_{J(\mathbf{f}),\delta}$ for all $i \in I_0$. There exist pairwise disjoint subsets $\{C_{i,\delta} : i \in I_0, \delta \in \{-1,1\}^{J(\mathbf{f})}\}$ of $\{-1,1\}^{K'_m}$, each of cardinality $2^{|K_m|-|J(\mathbf{f})|}$, such that $\psi_{J(\mathbf{f}),\delta,i,m} = 2^m \sum_{\eta\in C_{i,\delta}} \zeta_{\eta,m}$. Then

$$\|T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_c\|_{L^1} = \sum_{i\in I_0} \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} \sum_{\eta\in C_{i,\delta}} \frac{|a(i,\delta)c_\eta|}{2^{|K_m|}}.$$

Since $m \geq m_0(I_0, J(\mathbf{f}))$, $|I_0| \leq 2^m$, and $|J(\mathbf{f})| \leq |K_m|$. Choose subsets I_1 and J'_m of I such that $I_0 \cap I_1 = \emptyset$, $|I_0 \cup I_1| = 2^m$, $J(\mathbf{f}) \subseteq J'_m$, and $|J'_m| = |K_m| = |J_m|$. For $i \in I_0$, $\delta \in \{-1,1\}^{J(\mathbf{f})}$, there exists a bijection $\nu_{i,\delta} : C_{i,\delta} \to \{\tau \in \{-1,1\}^{J'_m} :$

$\tau_{|J(\mathbf{f})} = \delta\}$. Define $\tilde{h}_i = \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} \sum_{\eta\in C_{i,\delta}} c_\eta \phi_{J'_m,\nu_{i,\delta}(\eta)}$ for $i \in I_0$. Finally, there is a bijection

$$\beta\colon I_1 \times \{-1,1\}^{J'_m} \to \{-1,1\}^{K'_m} \backslash \cup \{C_{i,\delta} : i \in I_0, \delta \in \{-1,1\}^{J(\mathbf{f})}\}.$$

Define $\tilde{h}_i = \sum_{\tau\in\{-1,1\}^{J'_m}} c_{\beta(i,\tau)}\phi_{J'_m,\tau}$ for $i \in I_1$. Then let $\tilde{h}_i = 0$ if $i \notin I_0 \cup I_1$. It is straightforward to check that $\tilde{\mathbf{h}} = (\tilde{h}_i)_{i\in I}$ fulfills the requirements of the lemma. ∎

For all $m \in \mathbb{N}$, let B_m be the collection of all non-negative rational sequences $c = (c_\eta)_{\eta\in\{-1,1\}^{K'_m}}$ such that $\sum_{i\in I} \int |f_i h_{i,c}| \leq \|\mathbf{f}\|$ for all $\mathbf{f} = (f_i)_{i\in I} \in E$. Let us note that if $c \in B_m$, and $\tilde{\mathbf{h}} = (\tilde{h}_i)_{i\in I}$, $d_{\tilde{\mathbf{h}}} = d_{\mathbf{h}_c}$, then, due to the rearrangement invariance of the norm on E, $\sum_{i\in I} \int |f_i \tilde{h}_i| \leq \|\mathbf{f}\|$ for all $\mathbf{f} \in E$.

Proposition 14: *Let $\mathbf{f} = (f_i)_{i\in I} \in E$, and let I_0 be a finite subset of I such that $f_i = 0$ for all $i \notin I_0$. For all $m \geq m_0(I_0, J(\mathbf{f}))$,*

$$\sup_{c\in B_m} \|T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_c\|_{L^1} = \|\mathbf{f}\|.$$

Proof: By Lemma 13, for any $c \in B_m$, there exists $\tilde{\mathbf{h}} = (\tilde{h}_i)_{i\in I}$ such that $d_{\tilde{\mathbf{h}}} = d_{\mathbf{h}_c}$, and $\|T_{J(\mathbf{f}),m}\mathbf{f} \cdot h_c\|_{L^1} = \sum_{i\in I} \int |f_i \tilde{h}_i|$. The last sum is $\leq \|\mathbf{f}\|$ by the remark preceding the proposition. Conversely, for any $\varepsilon > 0$, there exists $x' \in E'$, $\|x'\| \leq 1$ such that $|x'(\mathbf{f})| > (1-\varepsilon)\|\mathbf{f}\|$. For $i_0 \in I_0$, and $\delta \in \{-1,1\}^{J(\mathbf{f})}$, let $\mathbf{x}_{i_0,\delta} = (x_i) \in E$, where $x_i = \phi_{J(\mathbf{f}),\delta}$ if $i = i_0$, and $x_i = 0$ otherwise. Set $b(i,\delta) = 2^{|J(\mathbf{f})|}x'(\mathbf{x}_{i,\delta})$ for $i \in I_0$, $\delta \in \{-1,1\}^{J(\mathbf{f})}$. Write $f_i = \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} a(i,\delta)\phi_{J(\mathbf{f}),\delta}$ for $i \in I_0$. Then

$$(1-\varepsilon)\|\mathbf{f}\| < |x'(\mathbf{f})| \leq \sum_{i\in I_0} \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} \frac{|a(i,\delta)b(i,\delta)|}{2^{|J(\mathbf{f})|}}.$$

Hence, there exist non-negative rational numbers $c(i,\delta)$ such that $c(i,\delta) \leq |b(i,\delta)|$, and

$$(1-\varepsilon)\|\mathbf{f}\| < \sum_{i\in I_0} \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} \frac{|a(i,\delta)|c(i,\delta)}{2^{|J(\mathbf{f})|}}.$$

Define $\mathbf{g} = (g_i)_{i\in I}$ by $g_i = \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} c(i,\delta)\phi_{J(\mathbf{f}),\delta}$ for $i \in I_0$, $g_i = 0$ otherwise. If $\mathbf{p} = (p_i)_{i\in I} \in E$, define $P_{J(\mathbf{f})}\mathbf{p} = (q_i)_{i\in I}$,

$$q_i = \sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} 2^{|J(\mathbf{f})|} \int p_i\phi_{J(\mathbf{f}),\delta} \cdot \phi_{J(\mathbf{f}),\delta}.$$

By a standard argument, using the rearrangement invariance of the norm on E, we see that $\|P_{J(\mathbf{f})}\mathbf{p}\| \leq \|\mathbf{p}\|$. Hence

$$\sum_{i\in I_0}\int |p_i g_i| \leq |x'|(P_{J(\mathbf{f})}|\mathbf{p}|) \leq \|\mathbf{p}\|. \tag{7}$$

From the proof of Lemma 13, there are pairwise disjoint subsets $\{C_{i,\delta} : i \in I_0, \delta \in \{-1,1\}^{J(\mathbf{f})}\}$ of $\{-1,1\}^{K'_m}$, each of cardinality $2^{|K_m|-|J(\mathbf{f})|}$, such that if we let $c_\eta = c(i,\delta)$ for all $\eta \in C_{i,\delta}$, $i \in I_0, \delta \in \{-1,1\}^{J(\mathbf{f})}$, and $c_\eta = 0$ otherwise, then for $c = (c_\eta)_{\eta\in\{-1,1\}^{K'_m}}$,

$$\|T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_c\|_{L^1} = \sum_{i\in I_0}\sum_{\delta\in\{-1,1\}^{J(\mathbf{f})}} \frac{|a(i,\delta)|c(i,\delta)}{2^{|J(\mathbf{f})|}} > (1-\varepsilon)\|\mathbf{f}\|.$$

Note that $d_{\mathbf{h}_c} = d_{\mathbf{g}}$. It follows from (7) that $\sum_{i\in I_0}\int |p_i h_{i,c}| \leq \|\mathbf{p}\|$ for all $\mathbf{p} = (p_i)_{i\in I} \in E$. Thus $c \in B_m$. Since $\varepsilon > 0$ is arbitrary, we obtain the reverse inequality

$$\sup_{c\in B_m} \|T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_c\|_{L^1} \geq \|\mathbf{f}\|.$$

This completes the proof the proposition. ∎

We are now ready to prove Theorem 8. For each $m \in \mathbb{N}$, B_m is countable. Hence we can list the functions $\{h_c : c \in B_m\}$ as a sequence $(h_{mn})_{n=1}^\infty$. Define the map $T: E \to \ell^\infty(\ell^\infty(L^1))$ by $T\mathbf{f} = (T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_{mn})_{mn}$. By Proposition 12, QT is a lattice homomorphism, where $Q: \ell^\infty(\ell^\infty(L^1)) \to \ell^\infty(\ell^\infty(L^1))/c_0(\ell^\infty(L^1))$ is the quotient map. It follows from Proposition 14 that QT is an (into) isometry. Finally, note that in the notation of Lemma 13 and Proposition 14,

$$T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_c \in \operatorname{span}\{\zeta_{\eta,m} : \eta \in \{-1,1\}^{K'_m}\}$$

for all $c \in B_m$, $m \geq m_0(I_0, J(\mathbf{f}))$. Hence

$$\|T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_c\|_{L^\infty} \leq 2^{|K'_m|}\|T_{J(\mathbf{f}),m}\mathbf{f}\cdot h_c\|_{L^1}.$$

Thus $QT\mathbf{f} \in QF_M$, where $M = (M_{mn})$, $M_{mn} = 2^{|K'_m|}$ for all m and n. An appeal to Theorem 3 yields the desired result.

3. Order isometry

Following [5], we say that a linear operator T from a Banach lattice E into a Banach lattice F is an **order isometry** if $Tx \geq 0$ if and only if $x \geq 0$, and $\|Tx\| = \|x\|$ for all $x \in E$. In [5], it is shown that if E is a separable Banach lattice, and E' has a weak order unit, then E' is order isometric to a closed subspace of $\overline{W}$. Here, we show that the assumption that E' has a weak order unit can be removed.

Let $\Gamma = \{-1,1\}^{\mathbb{N}}$. If $m \in \mathbb{N}$, and $\phi \in \Phi_m = \{-1,1\}^m$, let

$$\Gamma_\phi = \{\gamma \in \Gamma : \gamma_{|\{1,\ldots,m\}} = \phi\}.$$

PROPOSITION 15: *There is an order isometry from $\ell^\infty(\ell^1(\Gamma))$ onto a closed subspace of $(\bigoplus \ell^1(\Phi_m))_{\ell^\infty}/(\bigoplus \ell^1(\Phi_m))_{c_0}$.*

Proof: Partition $\mathbb{N}$ into a sequence of infinite subsets $(L_n)_{n=1}^\infty$. If $a \in \ell^\infty(\ell^1(\Gamma))$, write $a = (a_\gamma^n)$, so that $\|a\| = \sup_n \sum_{\gamma\in\Gamma} |a_\gamma^n| < \infty$. Given $m \in \mathbb{N}$, and $\phi \in \Phi_m$, define $b_\phi = \sum_{\gamma\in\Gamma_\phi} a_\gamma^n$, where n is the unique integer such that $m \in L_n$. Define the map U: $\ell^\infty(\ell^1(\Gamma)) \to (\bigoplus \ell^1(\Phi_m))_{\ell^\infty}$ by $Ta = b$, where $b = ((b_\phi)_{\phi\in\Phi_1}, (b_\phi)_{\phi\in\Phi_2}, \ldots)$. Clearly T is a positive linear operator. Note that if $m \in L_n$, then

$$\sum_{\phi\in\Phi_m} |b_\phi| \le \sum_{\phi\in\Phi_m} \sum_{\gamma\in\Gamma_\phi} |a_\gamma^n| = \sum_{\gamma\in\Gamma} |a_\gamma^n| \le \|a\|.$$

Hence $\|T\| \le 1$. Let Q: $(\bigoplus \ell^1(\Phi_m))_{\ell^\infty} \to (\bigoplus \ell^1(\Phi_m))_{\ell^\infty}/(\bigoplus \ell^1(\Phi_m))_{c_0}$ be the quotient map. Then QT is positive, and $\|QT\| \le 1$. We claim that QT is an order isometry.

If $QTa = Qb \ge 0$, then $\lim_{m\to\infty} \sum\{b_\phi : \phi \in \Phi_m, b_\phi \le 0\} = 0$. If $a \not\ge 0$, then there exist n_0 and γ_0 such that $a_{\gamma_0}^{n_0} < 0$. List the elements of L_{n_0} in ascending order: $L_{n_0} = \{m_1 < m_2 < \cdots\}$. For all $r \in \mathbb{N}$, let $\phi_r = \gamma_{0|\{1,\ldots,m_r\}}$. Then

$$\lim_{r\to\infty} b_{\phi_r} = \lim_{r\to\infty} \sum_{\gamma\in\Gamma_{\phi_r}} a_\gamma^{n_0} = a_{\gamma_0}^{n_0} < 0.$$

Thus

$$\lim_{r\to\infty} \sum_{\substack{\phi\in\Phi_{m_r} \\ b_\phi \le 0}} b_\phi \le a_{\gamma_0}^{n_0} < 0,$$

a contradiction. Therefore, $a \ge 0$.

Now, assume $\|a\| > 1$. Then there exists n such that $\sum_{\gamma\in\Gamma} |a_\gamma^n| > 1$. Given $\varepsilon > 0$, choose a finite subset Γ_1 of Γ such that

$$\sum_{\gamma\in\Gamma_1} |a_\gamma^n| > 1 \quad \text{and} \quad \sum_{\gamma\notin\Gamma_1} |a_\gamma^n| < \varepsilon.$$

Choose $m \in L_n$ so that if we define $\phi_\gamma = \gamma_{|\{1,\ldots,m\}}$ for all $\gamma \in \Gamma$, then $\phi_\gamma \neq \phi_{\gamma'}$ for all $\gamma, \gamma' \in \Gamma_1$, $\gamma \neq \gamma'$. For $\tilde{\gamma} \in \Gamma_1$,

$$|b_{\phi_{\tilde{\gamma}}}| = |\sum_{\gamma \in \Gamma_{\phi_{\tilde{\gamma}}}} a^n_\gamma| \geq |a^n_{\tilde{\gamma}}| - \sum_{\substack{\gamma \notin \Gamma_1 \\ \gamma \in \Gamma_{\phi_{\tilde{\gamma}}}}} |a^n_\gamma|.$$

Therefore,

$$\sum_{\gamma \in \Gamma_1} |b_{\phi_\gamma}| \geq \sum_{\gamma \in \Gamma_1} |a^n_\gamma| - \sum_{\gamma \notin \Gamma_1} |a^n_\gamma| > 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\|QTa\| \geq \|a\|$. Since $\|QT\| \leq 1$ as well, we conclude that QT is an isometry. ∎

LEMMA 16: *Let E be a separable Banach lattice. Then E' is isometrically lattice isomorphic to a sublattice of $\ell^\infty(\ell^1(\Gamma, L^1))$.*

Proof: By the proof of Lemma 3 in [5], for any $x \in E$, $x > 0$, there exist a compact Hausdorff space K, and a lattice homomorphism $S: C(K) \to E$ such that S' is a lattice homomorphism, $[0, S'x']$ is weakly (and hence norm) separable, and $\|S'x'\| = |x'|(x)$ for all $x' \in E'$. Note that E', and hence $S'E'$, has a dense subset of cardinality $\leq |\Gamma|$. Since $S'E'$ is a sublattice of the AL-space $M(K)$, has separable order intervals, and has density $\leq |\Gamma|$, it follows that $S'E'$ is isometrically lattice isomorphic to a sublattice of $\ell^1(\Gamma, L^1)$. Now let (x_n) be a positive sequence in the unit ball of E such that $\|x'\| = \sup_n |x'|(x_n)$ for all $x' \in E'$. For each n, there exists a lattice homomorphism $R_n: E' \to \ell^1(\Gamma, L^1)$ such that $\|R_n x'\| = |x'|(x_n)$ for all $x' \in E'$. Clearly, the map $R: E' \to \ell^\infty(\ell^1(\Gamma; L^1))$ defined by $Rx' = (R_n x')_{n=1}^\infty$ is an isometric lattice isomorphism. ∎

THEOREM 17: *Let E be a separable Banach lattice. Then E' is order isometric to a closed subspace of W.*

Proof: For any $n \in \mathbb{N}$, let E_n be the conditional expectation operator on L^1 with respect to the σ-algebra generated by the sets $\{[(k-1)/2^n, k/2^n) : 1 \leq k \leq 2^n\}$. Then the map $V: \ell^1(\Gamma, L^1) \to (\bigoplus_{n=1}^\infty \ell^1(\Gamma, E_n L^1))_{\ell_\infty}$ defined by $V((f_\gamma)_{\gamma\in\Gamma}) = ((E_n f_\gamma)_{\gamma\in\Gamma})_{n=1}^\infty$ is an order isometry. Since $\ell^1(\Gamma, E_n L^1)$ is clearly isometrically lattice isomorphic to $\ell^1(\Gamma)$, it follows that $\ell^\infty(\ell^1(\Gamma, L^1))$, and hence E', is order isometric to a closed subspace of $\ell^\infty(\ell^1(\Gamma))$, which in turn is order isometric to a closed subspace of $(\bigoplus \ell^1(\Phi_m))_{\ell^\infty}/(\bigoplus \ell^1(\Phi_m))_{c_0}$ by Proposition 15. It is a simple exercise to check that the latter space is isometrically lattice isomorphic to a sublattice of QF_M for a suitably chosen $M = (M_{ij})$. Finally, QF_M is isometrically lattice isomorphic to a sublattice of W by Theorem 3. ∎

References

[1] M. Cwikel and C. Fefferman, *Maximal seminorms on WeakL^1*, Studia Mathematica **69** (1980), 149–154.

[2] M. Cwikel and C. Fefferman, *The canonical seminorm on WeakL^1*, Studia Mathematica **78** (1984), 275–278.

[3] J. Kupka and N. T. Peck, *The L_1 structure of weak L_1*, Mathematische Annalen **269** (1984), 235–262.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, Berlin, 1979.

[5] H. P. Lotz and N. T. Peck, *Sublattices of the Banach envelope of WeakL^1*, Proceedings of the American Mathematical Society **126** (1998), 75–84.

[6] N. T. Peck and M. Talagrand, *Banach sublattices of weak L_1*, Israel Journal of Mathematics **59** (1987), 257–271.

[7] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin, 1974.